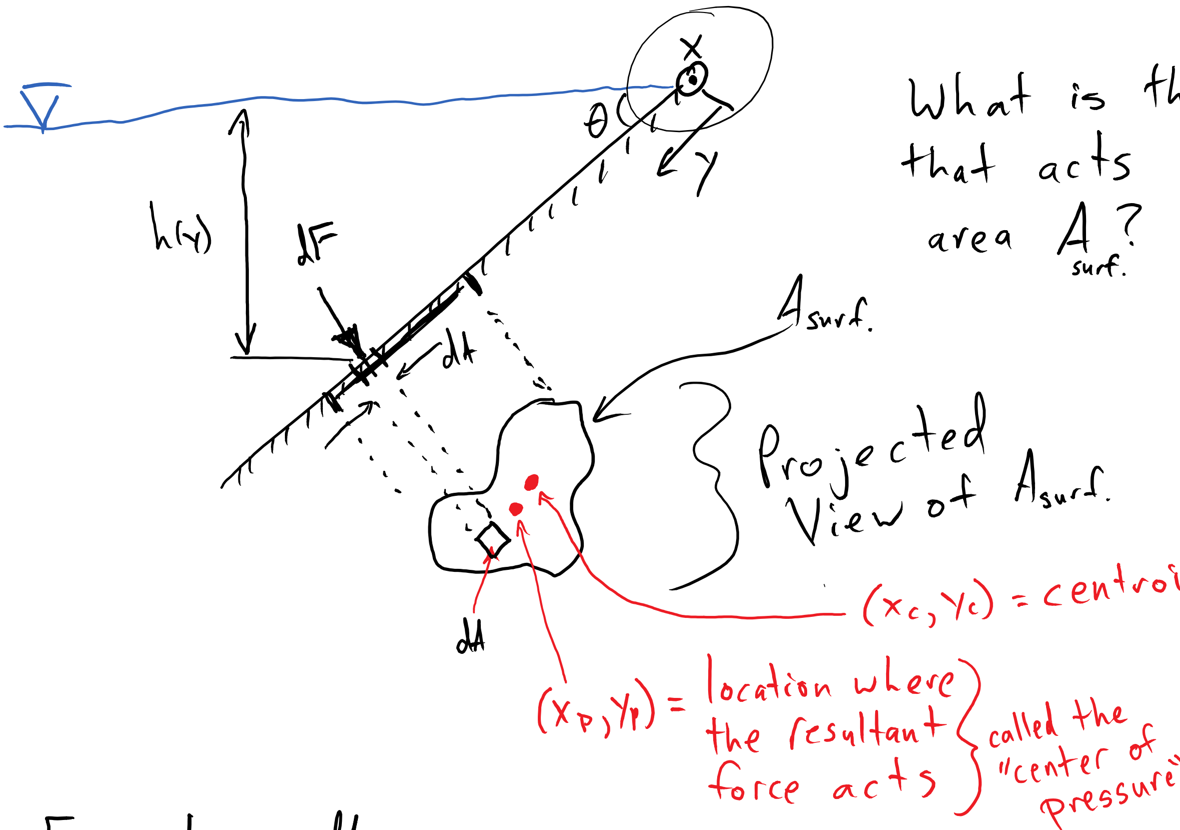
Forces on Inclined Surfaces of arbitrary shape

What is the force that acts on the area $A_{surf.}$?



Force acting on dA:

$$dF = p(y)\cdot dA$$

Total force: $$F_R = \int_{A_{surf}} p(y)\cdot dA = \int_{A_{surf}} (p_o + \underbrace{h(y)}_{y\cdot\sin\theta}\cdot \underbrace{\rho\cdot g}_{\gamma})\cdot dA$$

$\gamma = \rho\cdot g$ is the specific weight of the fluid

$$= \int_{A_{surf}} p_o\cdot dA + \gamma\cdot\sin\theta \underbrace{\int_{A_{surf}} y\cdot dA}_{y_c\cdot A_{surf}}$$

$$= p_o\cdot A_{surf} + \gamma\cdot\sin\theta\cdot y_c\cdot A_{surf}$$

$$\boxed{F_R = (p_o + \gamma\cdot y_c\cdot\sin\theta)\cdot A_{surf}}$$

definition of the centroid

$$y_c = \frac{\int_{A_{surf}} y\cdot dA}{\int_{A_{surf}} dA} = \frac{\int_{A_{surf}} y\cdot dA}{A_{surf}}$$

$$\Rightarrow \int_{A_{surf}} y\cdot dA = y_c\cdot A_{surf.}$$

-OR- $\boxed{F_R = (p_0 + \gamma \cdot h_c) \cdot A_{surf}}$

h_c is the depth of the centroid of the surface

Note: $\underset{\sim}{F_R}$ does not generally act at h_c!

It acts at (x_p, y_p).

Find (x_p, y_p):

$$(\Sigma M)_x \Rightarrow F_R \cdot y_p = \int_{A_{surf.}} y \overbrace{(p_0 + \gamma \cdot y \cdot \sin\theta)\, dA}^{dF}$$

$$= \int_{A_{surf}} p_0 \cdot y \cdot dA + \int_{A_{surf}} \gamma \cdot y^2 \cdot \sin\theta \cdot dA$$

$$= p_0 \int_{A_{surf}} y \cdot dA + \gamma \sin\theta \int_{A_{surf}} y^2\, dA$$

$$= p_0 \cdot y_c \cdot A_{surf} + \gamma \cdot \sin\theta \cdot \overbrace{(\underbrace{I_0}_{\substack{\text{2nd moment} \\ \text{of area} \\ \text{about the} \\ \text{centroid } y_c}} + \underbrace{y_c^2 A_{surf}}_{\text{parallel-axis theorem}})}^{I_{xx}}$$

$$F_R \cdot y_p = p_0 \cdot y_c \cdot A_{surf} + \gamma \cdot \sin\theta \cdot (I_0 + y_c^2 \cdot A_{surf})$$

$$y_p = \frac{p_0 \cdot y_c \cdot A_{surf} + \gamma \cdot \sin\theta \cdot (I_0 + y_c^2 A_{surf})}{(p_0 + \gamma \cdot y_c \cdot \sin\theta)\, A_{surf}}$$

$$= \frac{(p_0 \cdot y_c \cdot A_{surf}) + \gamma \cdot \sin\theta \cdot I_x + (\gamma \cdot \sin\theta\, y_c^2 \cdot A_{surf})}{\underbrace{(p_0 + \gamma \cdot y_c \cdot \sin\theta)}_{p_c} \cdot A_{surf}}$$

p_c = pressure at y_c

$$= \frac{\overbrace{(p_0 + \gamma \cdot \sin\theta \cdot y_c)}^{p_c} \cdot y_c \cdot A_{surf} + \gamma \cdot \sin\theta \cdot I_0}{p_c \cdot A_{surf}}$$

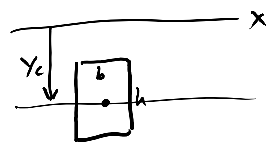
2nd moment of

$p_c \cdot A_{surf}$

$$\boxed{y_p = y_c + \frac{\gamma \cdot \sin\theta \cdot I_o}{p_c \cdot A_{surf}}}$$

I_o is 2nd moment of area of A_{surf} about an axis ∥ to x-axis, through y_c.

e.g. if $A_{surf} = b \cdot h$, then I_o is $\frac{b \cdot h^3}{12}$



If $p_0 = 0$ (gage pressure is the atmospheric pressure)

Then $$\boxed{y_p = y_c + \frac{I_o}{y_c \cdot A_{surf}}}$$

Similarly, x_p: $\quad \left(\sum M\right)_y \Rightarrow F_R \cdot x_p = \int_{A_{surf}} x \cdot p(y) \cdot dA$

⋮

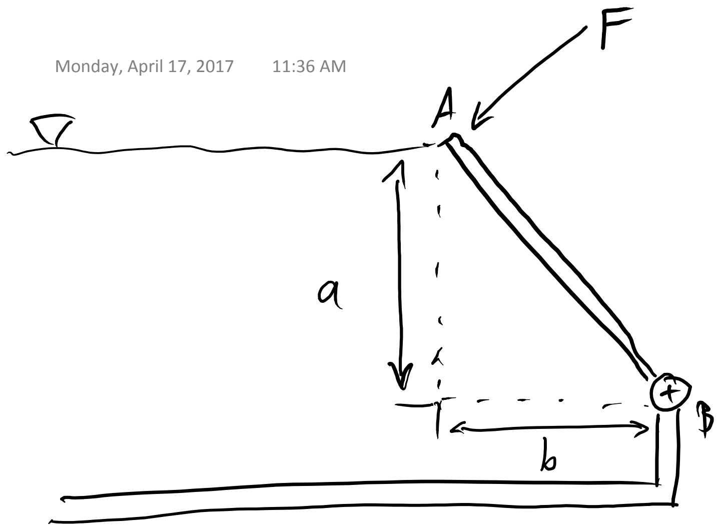
Algebra

⋮

only needed if A_{surf} is asymmetric
$$\left\{ \begin{array}{l} x_p = \dfrac{I_{xy}}{y_c \cdot A_{surf}} \quad \text{if } p_0 = 0 \\ \& \quad I_{xy} = \displaystyle\int_{A_{surf}} x \cdot y \cdot dA \end{array} \right.$$

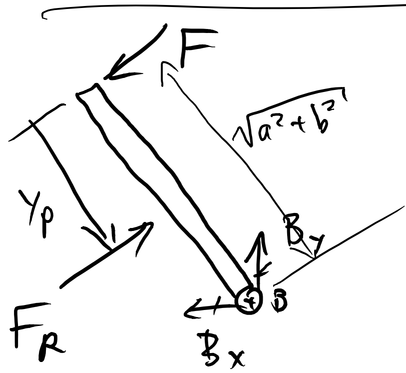
Summary (if $p_0 = 0$)

$F_R = p_c \cdot A_{surf}$ acts at
$$\left\{ \begin{array}{l} x_p = \dfrac{I_{xy}}{y_c \cdot A_{surf}} \\ (I_{xy} = 0 \text{ if symmetric}) \\ y_p = y_c + \dfrac{I_o}{y_c \cdot A_{surf}} \end{array} \right.$$



Find F to hold the gate in this position (gate has with w)

FBD of Gate



$$(\Sigma M)_B = 0$$

$$F\cdot\sqrt{a^2+b^2} - F_R\cdot\left(\sqrt{a^2+b^2} - y_p\right) = 0$$

$$F = F_R\cdot\left(1 - \frac{y}{\sqrt{a^2+b^2}}\right)$$

$$p_c = \gamma\cdot\frac{a}{2}\ , \quad A_{surf} = \sqrt{a^2+b^2}\cdot w$$

$$F = \underbrace{\frac{\gamma\cdot a\cdot w}{2}\sqrt{a^2+b^2}}_{F_R = p_c\cdot A_{surf}}\cdot\left(1 - \frac{y_p}{\sqrt{a^2+b^2}}\right) = \frac{\gamma\cdot a\cdot w}{2}\left(\sqrt{a^2+b^2} - y_p\right)$$

$$y_p = y_c + \frac{I_o}{y_c\cdot A_{surf}} \quad \left\{ \begin{aligned} I_o &= \frac{w\cdot\left(\sqrt{a^2+b^2}\right)^3}{12} \\ A_{surf} &= w\sqrt{a^2+b^2} \\ y_c &= \frac{1}{2}\sqrt{a^2+b^2} \end{aligned} \right.$$

$$y_p = \frac{1}{2}\sqrt{a^2+b^2} + \frac{\frac{\cancel{w}\left(\cancel{\sqrt{a^2+b^2}}\right)^{\cancel{3}}}{12}}{\frac{1}{2}\cancel{\sqrt{a^2+b^2}}\cdot\cancel{w}\cdot\cancel{\sqrt{a^2+b^2}}} = \sqrt{a^2+b^2}\cdot\left(\frac{1}{2} + \frac{1}{6}\right)$$

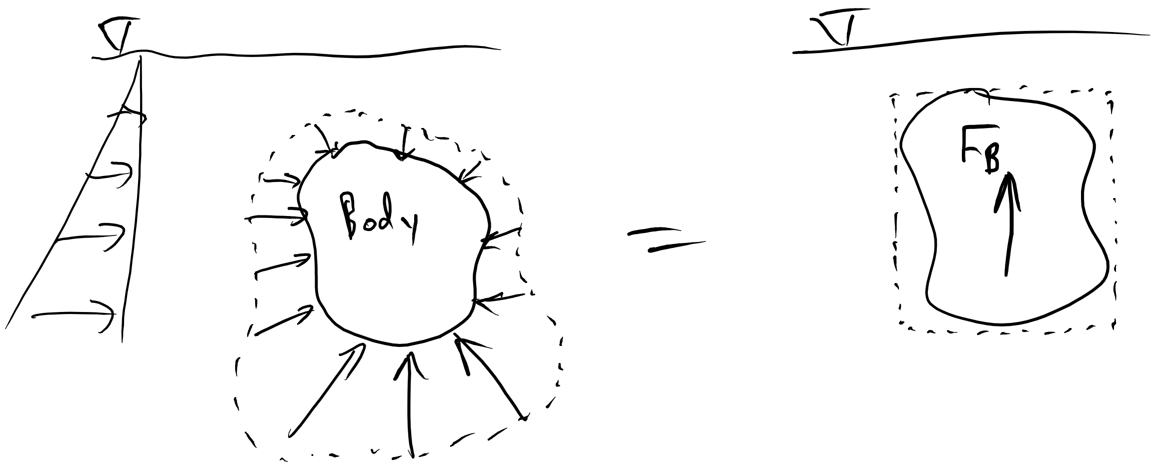
$$\ldots 2\sqrt{a^2+b^2} \ldots \qquad \frac{\ldots}{\frac{1}{2}\cancel{\sqrt{a^2+b^2}}\cdot\cancel{w}\cdot\cancel{\sqrt{a^2+b^2}}} = \sqrt{a^2+b^2}\cdot\left(\frac{\ldots}{2} + \frac{\ldots}{6}\right)$$

$$Y_p = \sqrt{a^2+b^2}\cdot\frac{2}{3}$$

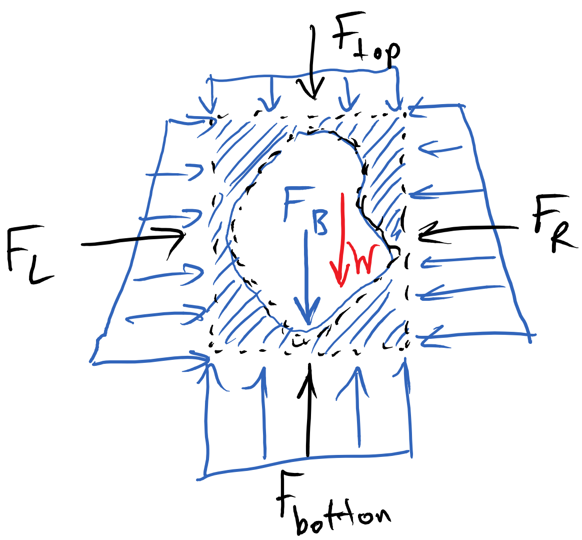
$$F = \frac{\gamma\cdot a\cdot w}{2}\left[\sqrt{a^2+b^2} - \frac{2}{3}\sqrt{a^2+b^2}\right]$$

$$\boxed{F = \frac{\gamma\cdot a\cdot w}{6}\sqrt{a^2+b^2}}$$

Bouyancy (Archimedes Principle)



Find F_B
"Buoyant force"



F_B is the force of the body on the shaded water

W is the weight of the shaded water